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1997 J. Phys. A: Math. Gen. 30 733

(http://iopscience.iop.org/0305-4470/30/2/033)

J. Phys. A: Math. Gen. **30** (1997) 733–753. Printed in the UK PII: S0305-4470(97)74459-7

Phase-space structure of a new integrable system related to hydrogen atoms in external fields

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Received 14 May 1996

Abstract. We explore the detailed topological structure of the phase-space of the recently discovered three-dimensional integrable but nonseparable Hamiltonian system with velocity dependent potential. Two three-parameter families of three-dimensional tori which foliate the phase-space are identified. The complete classification, according to their topology, of the level sets corresponding to the critical points of the energy-momentum map, is accomplished. The relationship of the three-dimensional integrable system with important physical systems of current interest, namely, the hydrogen atoms in circularly polarized (CP) fields, in crossed magnetic and electric fields and in crossed magnetic and CP fields, is discussed.

1. Introduction

In a recent paper, Raković and Chu (1995a), we have shown that the three-dimensional dynamical system defined with the Hamiltonian function with velocity dependent potential:

$$H = \frac{\boldsymbol{p}^2}{2} - \frac{1}{r} - \omega l_z + fx + \frac{\omega^2}{18}(r^2 + 3z^2) \tag{1}$$

is integrable. The integrability follows from the existance of two additional independent integrals of motion which are in involution:

$$\begin{aligned} H_1 = {} & \frac{4}{3}\omega l_z(H - fx) + \frac{8}{9}\omega^2 l_z^2 + f\left(l_y p_z - l_z p_y + \frac{x}{r}\right) + \frac{f^2}{2}(y^2 + z^2) \\ & + \frac{1}{3}\omega f(3xl_z - p_y r^2 + zl_x) - \frac{1}{9}\omega^2 fx(r^2 + z^2) \end{aligned} \tag{2}$$

$$H_2 = \omega\left(l_x p_y - l_y p_x + \frac{z}{r}\right) + \frac{1}{9}\omega^3 z(x^2 + y^2) - \frac{3}{2}fl_x - \frac{1}{2}\omega fxz. \tag{3}$$

This dynamical system is a rare example of an integrable but not separable Hamiltonian system which is, at the same time, related to important realistic systems: hydrogen atoms in circularly polarized (CP) microwave fields and hydrogen atoms in crossed magnetic and electric fields. Our goal in this paper is to explore the topological structure of the phase-space which corresponds to the Hamiltonian function H, or more precisely, to decompose the phase-space into irreducible sets invariant under motions generated by all three functions, equations (1)–(3), treated as Hamiltonian functions.

The functions H, H_1 and H_2 formally depend on two external parameters, f and ω, however, the topological structure of the phase-space depends only on a certain combination of the parameters due to the following scaling property. If one introduces the scaled canonical coordinates $\boldsymbol{r}_s = \omega^{2/3}\boldsymbol{r}$, $\boldsymbol{p}_s = \omega^{-1/3}\boldsymbol{p}$ and the parameter $f_s = \omega^{-4/3} f$, then the scaled Hamiltonian functions $H_s = \omega^{-2/3} H$, $H_{1s} = \omega^{-4/3} H_1$ and $H_{2s} = \omega^{-1} H_2$ have the same functional forms as in equations (1)–(3) with the replacement $\omega = 1$. Therefore, the topological structure of the phase-space depends only on the scaled parameter $f_s = \omega^{-4/3} f$ (and not independently on f and ω). Throughout this paper we shall only use scaled variables (or equivalently we set $\omega = 1$ and $f = f_s$) while omitting the subscript s.

The plan of the paper is the following. In section 2 we recall some basic mathematical notions. In section 3 we study important reduced two-dimensional system which describes the motion in the plane $z = 0$, while in section 4 we study the full three-dimensional system. Finally, in section 5, we discuss the relationship between our integrable system and some realistic physical systems.

2. Some basic mathematical notions

To facilitate the discussion below, we shall first recall basic notions concerning integrable systems. We closely follow the book by Arnol'd (1978). More advanced treatment of integrable systems from algebraic and geometrical points of view can be found e.g. in the books by Perelomov (1990) and Fomenko (1988).

An n-dimensional dynamical system with the phase-space P, (local) canonical coordinates $(q, p) = (\xi_1, \ldots, \xi_{2n}) = \xi$ and the Hamiltonian function $H(\xi)$ is integrable if it possesses n independent integrals of motion $H_1(\xi), \ldots, H_n(\xi) = H(\xi)$ which are in involution. Each integral of motion–Hamiltonian function H_i, $i = 1, \ldots, n$, is a generator of the Hamiltonian vector field $X_i = (\partial H_i/\partial p, -\partial H_i/\partial q)^T$, and one-parameter group g_i^t of the phase-space transformations (diffeomorphisms). The composition of these groups $g^{t_1,\ldots,t_n} = g_1^{t_1} \circ \ldots \circ g_n^{t_n}$ is n-parameter group of the phase-space transformations (and defines an action of the Abelian group $\boldsymbol{R}^n$ on the phase-space P). The energy-momentum map is:

$$EM : P \to \boldsymbol{R}^n : (\xi_1, \ldots, \xi_{2n}) \to (H_1(\xi), \ldots, H_n(\xi)) \qquad EM(P) = Q \tag{4}$$

where $Q \subset \boldsymbol{R}^n$ is the image of the phase-space P under energy-momentum map. The phase-space decomposes into a disjoint union of the level sets of the energy-momentum map:

$$\begin{aligned} &P = \bigcup \mu_C \qquad C = (C_1, \ldots, C_n) \in Q \\ &\mu_C = \{\xi : H_i(\xi) = C_i, i = 1, \ldots, n\} = EM^{-1}(C). \end{aligned} \tag{5}$$

Each level set μ_C is invariant under the action of the group g, but (in general) as not irreducible, i.e. it further decomposes into a disjoint union of orbits of the group g:

$$\mu_C = \bigcup O_\xi \qquad \xi \in \mu_C \qquad O_\xi = \{g^{t_1,\ldots,t_n}(\xi) : -\infty < t_i < \infty\}. \tag{6}$$

The orbits O_ξ are the minimal invariant sets (smooth manifolds) under the action of the group g. The dimension of the orbit is determined by the rank of the differential of the energy-momentum map:

$$\dim(O_\xi) = \mathrm{rank}(dEM(\xi)) = \mathrm{rank}(M(\xi)) \leqslant n \qquad M = [\partial H_i/\partial \xi_j]_{n\times 2n}. \tag{7}$$

According to the Liouville's theorem, if $\dim(O_\xi) = \mathrm{rank}(M(\xi)) = k \leqslant n$, then, for some $k_1 \leqslant k$

$$O_\xi \cong S^1 \times \cdots \times S^1 \times \boldsymbol{R} \times \cdots \times \boldsymbol{R} \cong T^{k_1} \times \boldsymbol{R}^{k-k_1}. \tag{8}$$

In particular, if the orbit is compact then $O_\xi \cong T^k$, i.e. the orbit is diffeomorphic to k-dimensional torus.

The point C is said to be *regular* if it is a regular value of the energy-momentum map EM, i.e. if at each $\xi \in \mu_C$, vector fields X_i are linearly independent or equivalently, $\mathrm{rank}(M(\xi)) = n$. If the corresponding set μ_C is disconnected then each of its connected components is an n-dimensional orbit of the group g. The point C is *critical* if at some phase-space point(s) (belonging to μ_C) vector fields X_i are linearly dependent, i.e. in the decomposition (6) of the level set μ_C of the critical point, at least one of the orbits consists of the phase-space points for which $\mathrm{rank}(M(\xi)) = k < n$ and is therefore k-dimensional. We shall say that such an orbit is also critical.

3. Phase-space structure of reduced two-dimensional system

The Hamiltonian function H, equation (1), possesses one geometrical symmetry. It is invariant under reflections with respect to the four-dimensional plane $z = 0, \quad p_z = 0$. From the equations of motion one easily verifies that the same plane is invariant under the motion generated by the Hamiltonian functions H and H_1, equations (1) and (2). (From equation (3) it follows that this plane corresponds to the zero value of the Hamiltonian function H_2: $z = 0$, $p_z = 0 \Rightarrow H_2 = 0$.) This leads to the definition of the (reduced) two-dimensional integrable system with two commuting and independent Hamiltonian functions:

$$H' = \frac{p_x^2 + p_y^2}{2} - \frac{1}{(x^2 + y^2)^{1/2}} - l_z + f_s x + \frac{1}{18}(x^2 + y^2) \tag{9}$$

$$\begin{aligned} H_1' = {}& \frac{4}{3} l_z (H' - f_s x) + \frac{8}{9} l_z^2 + f_s \left(-l_z p_y + \frac{x}{(x^2 + y^2)^{1/2}} \right) + \frac{1}{2} f_s^2 y^2 \\ & + \frac{f_s}{3}(3x l_z - p_y(x^2 + y^2)) - \frac{f_s}{9} x (x^2 + y^2). \end{aligned} \tag{10}$$

In this section we shall study this reduced two-dimensional system. It is a good starting point for understanding of the full three-dimensional system to which we shall return in the next section. Anyway, the Hamiltonian function H' corresponds to a special case of an interesting realistic system (see section 5).

In this case the phase-space is $P' = \{(x, y, p_x, p_y)\}$, its image under the energy-momentum map is $Q' = EM(P')$, Q'_r and $Q'_{cr} = Q' \setminus Q'_r$ are the sets of all regular and critical points respectively. The point $(E, C_1) \in Q'$ (E and C_1 are fixed values of H' and H'_1) is critical if the corresponding level set μ_{E,C_1} contains at least one critical orbit of the two-parameter group g' (generated by H' and H'_1) of a lower dimension than two. If the point (E, C_1) is regular then μ_{E,C_1} is a disjoint union of a finite number of two-dimensional smooth manifolds—orbits of the group g'.

3.1. Critical points

Shown in figures 1(a) and (b) are the plots of the set Q'_{cr} of critical points for two different values of the parameter $f_s = 2.67$ and $f_s = 0.5$. It appears that these are only two topologically different cases, i.e. when $f_s > (\frac{4}{9})^{2/3}$ the plots of critical points are homeomorphic to the one in figure 1(a), while if $f_s \leqslant (\frac{4}{9})^{2/3}$ they are homeomorphic to the plot in figure 1(b). In the first case the set Q'_r of regular points consists of four open connected components Q'_1, Q'_2, Q'_3 and Q'_4 while in the second case only the first three exist. The shaded regions in figures 1(a) and (b) do not belong to Q'. It is convenient to distinguish two points F_+ and F_- in Q'_{cr} and to decompose the rest of the set into nine curves $\theta_1^-, \theta_2^-, \epsilon^+, \theta_1^+, \theta_2^+, \theta_3^+, \theta_4^+, \epsilon_2^-$ and ϵ_1^- in the first case (figure 1(a)) or into six

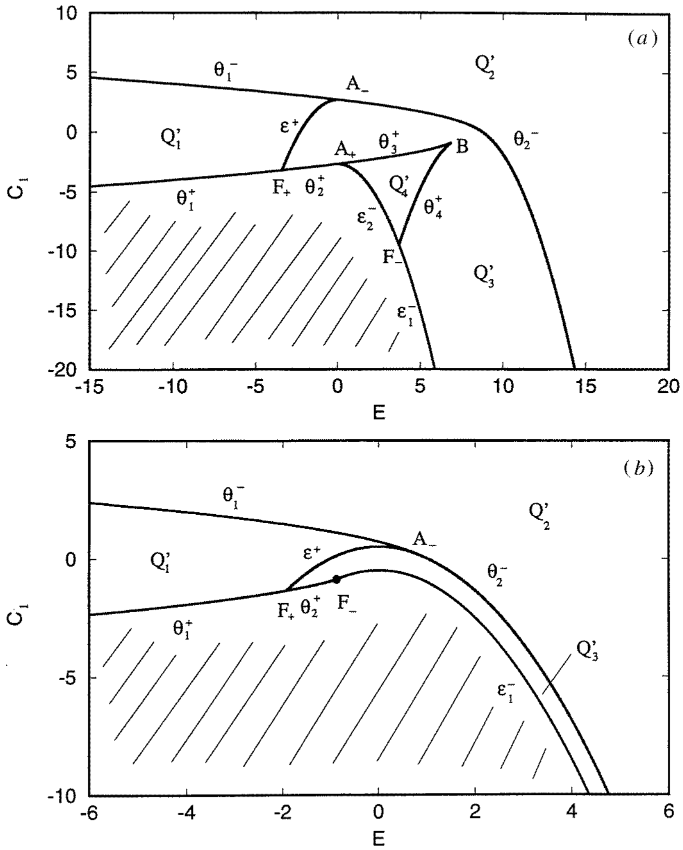

Figure 1. The plots of the sets of critical points Q'_{cr} for (a) $f_s = 2.67$ and (b) $f_s = 0.5$.

curves θ_1^-, θ_2^-, ϵ^+, θ_1^+, θ_2^+ and ϵ_1^- in the second case (figure 1(b)). By definition the points A_-, A_+ and B belong to θ_2^-, θ_2^+ and θ_3^+ resepectively. This decomposition is dictated by the topological structure of the level sets, i.e. the level sets of critical points which belong to the same curve are mutually diffeomorphic. Analogously, the level sets of regular points which belong to the same component Q'_i of the set Q'_r are mutually diffeomorphic. We shall now give analytical expressions for the critical points and corresponding critical orbits of the group g'. Then we shall consider the level sets of the points, both regular and critical, that belong to different regions in figure 1(a).

3.2. Zero-dimensional orbits

The critical points F_+ and F_- correspond to only two fixed points (denoted also with $F_\pm$) of both Hamiltonian functions H' and H'_1, i.e. to zero-dimensional orbits of the group g'. Their locations in the phase-space P' are

$$x_{F_\pm} = \mp r_\pm \qquad y_{F_\pm} = 0 \qquad p_{xF_\pm} = 0 \qquad p_{yF_\pm} = x_{F_\pm} \tag{11}$$

where $r_\pm$ is the positive root of the cubic equation

$$8r^3/9 \pm f_s r^2 - 1 = 0. \tag{12}$$

The corresponding values of the Hamiltonian functions H' and H_1' are

$$E_{F_\pm} = (-3 \mp f_s {r_\pm}^2)/(2r_\pm) \qquad C_{1F_\pm} = -{f_s}^2 {r_\pm}^2/8 - r_\pm \mp 7 f_s/8. \tag{13}$$

In the limit of the static field when $f_s \to \infty$ (i.e. when $\omega \to 0$ and f is finite) the fixed point F_+ corresponds to the Stark saddle point while F_- becomes imaginary.

3.3. *One-dimensional orbits*

The curves $\theta_1^-, \theta_2^-, \theta_1^+, \theta_2^+, \theta_3^+$ and θ_4^+ are all determined by the same pair of equations:

$$E = -2\theta/3 + 9{f_s}^2/8 - 1/(2\theta^2) \qquad C_1 = -2\theta^2/9 - 2/(3\theta) \tag{14}$$

and each has its own range of the parametar θ:

$$\begin{aligned} &-\infty < \theta_2^- \leqslant \theta_{A_-} < \theta_1^- < 0 < \theta_1^+ < \theta_{F_+} \\ &\theta_{F_+} < \theta_2^+ \leqslant \theta_{A_+} < \theta_3^+ \leqslant \theta_B < \theta_4^+ < \theta_{F_-} \qquad \text{if } f_s > (\tfrac{4}{9})^{2/3} \\ &\theta_{F_+} < \theta_2^+ < \theta_{F_-} \qquad \text{if } f_s \leqslant (\tfrac{4}{9})^{2/3} \end{aligned} \tag{15}$$

where

$$\begin{aligned} &\theta_{F_\pm} = \mp 3 f_s r_\pm/4 - {r_\pm}^2/3 + \left((\mp 3 f_s r_\pm/4 - {r_\pm}^2/3)^2 + r_\pm\right)^{1/2} \\ &\theta_{A_\pm} = \pm 2/(3 f_s) \qquad \theta_B = (\tfrac{3}{2})^{1/3}. \end{aligned} \tag{16}$$

The triangle $A_+ B F_-$ exists only for $f_s > (\frac{4}{9})^{2/3}$. For $f_s = (\frac{4}{9})^{2/3}$ it contracts to a point F_-. The points belonging to these curves are critical, and the corresponding level sets contain one-dimensional critical orbits of the group g'. Given the value of the parameter θ, the locations in the phase-space P' of these orbits are given in polar coordinates ($x = \rho\cos\phi$, $y = \rho\sin\phi$) with:

$$\theta\rho^2 - \rho((\tfrac{9}{4}) f_s \theta \cos\phi + \tfrac{3}{2}) + 3\theta^2/2 = 0 \qquad p_\rho = (\tfrac{3}{2}) f_s \sin\phi \qquad p_\phi = \theta + \rho^2/3. \tag{17}$$

The last equation defines exactly one orbit diffeomorphic to circle when $\theta < 0$ or when $\theta_{F_+} < \theta < \theta_{F_-}$. For $\theta = \theta_{F_-}$ the circle contracts to the (stable) fixed point F_-, see equations (11)–(13). When $0 < \theta < \theta_{F_+}$ equation (17) defines two disjoint orbits-circles. All these one-dimensional critical orbits of the group g' are at the same time periodic orbits of the Hamiltonian function H'.

It appears that the level sets of the points from the curves θ_1^-, θ_2^-, θ_3^+ and θ_4^+ contain, besides critical also some two-dimensional orbits, while the level sets of the points from curves θ_1^+ and θ_2^+ consist of only critical one-dimensional orbits defined by equation (17). Two periodic orbits (i.e. the level set) corresponding to one of the critical points from the curve θ_1^+ are given in figure 2(*b*). The level sets of all other points from the curve θ_1^+ have the same topological structure. These two periodic orbits meet at the point of bifurcation F_+ which corresponds to the fixed point (of H') (see equations (11)–(13)), i.e. the solution of equation (17) for $\theta = \theta_{F_+}$ is represented by the disjoint union of one zero-dimensional orbit—unstable fixed point F_+, and two one-dimensional orbits (of the group g') diffeomorphic to real line $\boldsymbol{R}$. These two critical orbits constitute what is known as stable and unstable manifold of unstable fixed point F_+. The plot of the level set of critical point F_+ is given in figure 2(*c*). The plot of the level set—periodic orbit corresponding to one of the critical points from curve θ_2^+ is given in figure 3(*b*).

The curve ϵ^+ in figures 1(*a*) and (*b*) is determined with

$$C_1 = -E^2/2 + f_s \qquad E_{F_+} < E < E_{A_-} = 4/(9 f_s). \tag{18}$$

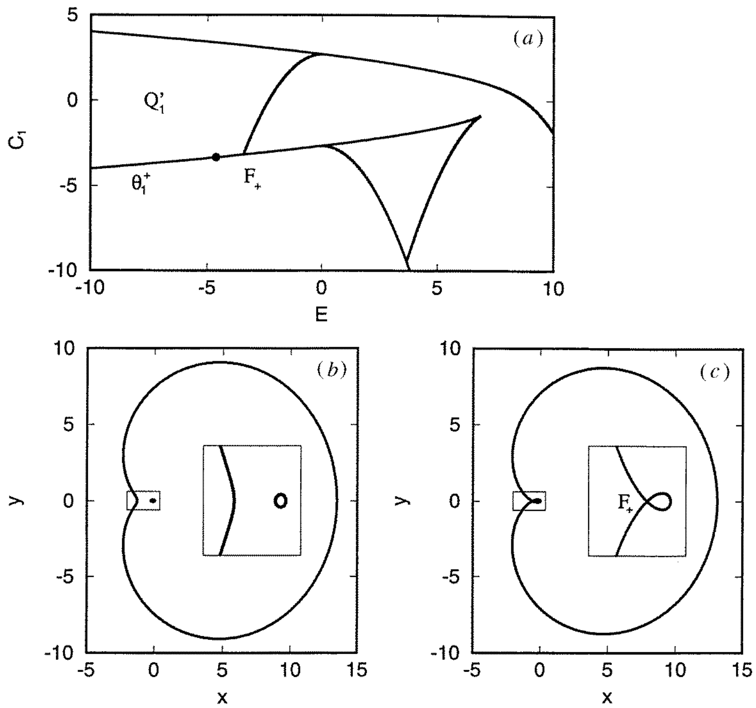

Figure 2. (b) The projection on the x–y plane of the level set of one of the critical points from the curve θ_1^+ whose location is shown in (a). (c) The same as in (b) for the critical point F_+.

For ϵ_1^- and ϵ_2^- we have

$$C_1 = -E^2/2 - f_s \qquad E_{A_+} = -4/(9f_s) < E_{\epsilon_2^-} < E_{F_-} < E_{\epsilon_1^-} < \infty. \tag{19}$$

Note that ϵ_2^- exists only when $f_s > (\frac{4}{9})^{2/3}$. Any given point from the curve ϵ^+ where energy E corresponds exactly to one one-dimensional critical orbit of the group g' diffeomorphic to circle (periodic orbit of the Hamiltonian functions H') whose location in the phase-space is given (again in polar coordinates) by:

$$\begin{aligned} &\pm(\tfrac{4}{3})(2\rho)^{1/2}(1 - f_s\rho^2)^{1/2}\sin(\phi/2) - 2f_s\rho\sin^2(\phi/2) + 4\rho^2/9 - E = 0 \\ &p_\rho = \pm(2(1 - f_s\rho^2)/\rho)^{1/2}\cos(\phi/2). \end{aligned} \tag{20}$$

The expression (rather complicated) for the momentum p_ϕ as a function of ρ, ϕ and p_ρ can easily be obtained from equations (9), (10) and (18).

Similarly, the one-dimensional orbits corresponding to critical points from the curves ϵ_1^- and ϵ_2^- are determined by the equations:

$$\begin{aligned} &\pm(\tfrac{4}{3})(2\rho)^{1/2}(1 + f_s\rho^2)^{1/2}\cos(\phi/2) + 2f_s\rho\cos^2(\phi/2) + 4\rho^2/9 - E = 0 \\ &p_\rho = \mp(2(1 + f_s\rho^2)/\rho)^{1/2}\sin(\phi/2). \end{aligned} \tag{21}$$

For each energy E from interval (E_{A_+}, E_{F_-}), i.e. for the points from ϵ_2^- the solution of the last equation coincides with the entire level set which consists of two disjoint periodic orbits of H'. The plot of the level set of one of the points from the curve ϵ_2^- is given in figure 3(c). For $E = E_{F_-}$ one of these circles contracts into the stable fixed point F_-, see figure 3(d), and for $E > E_{F_-}$ (curve ϵ_1^-) the solution of equation (21) is only one periodic orbit.

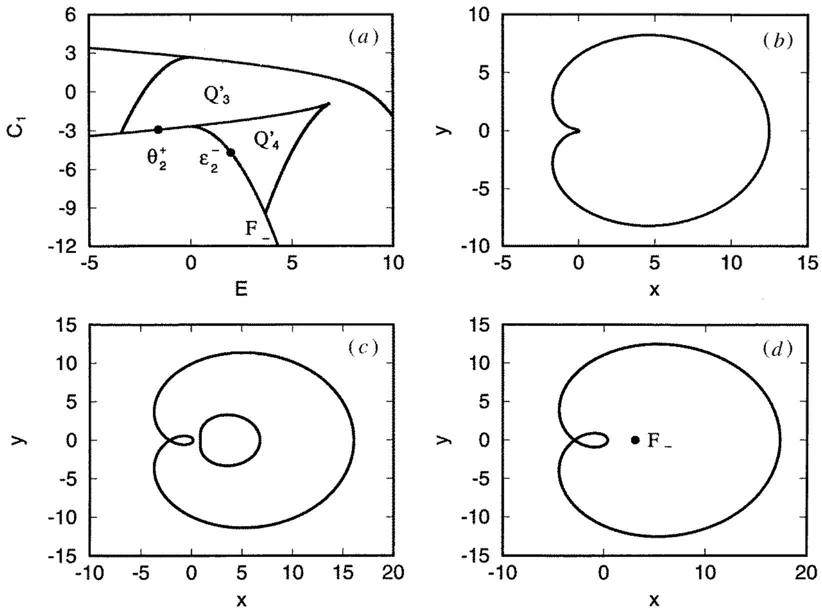

Figure 3. The projections on the x–y plane of the periodic trajectories of the Hamiltonian function H which correspond to the critical point: (b) from the curve θ_2^+, (c) from the curve ϵ_2^-, (d) F_-. The locations of the points are given in (a).

In summary, we note that the points F_+, F_-, A_+, A_- and B can all be seen as the bifurcation points of certain periodic orbits. For example, we see (figures 2 and 3) that two periodic orbits which constitute the level set of the critical point from θ_1^+ bifurcate to a single periodic orbit when corresponding critical point moves across F_+ into θ_2^+, or they bifurcate to another periodic orbit if the critical point moves into ϵ^+. The stability of these periodic orbits is discussed in the following subsection.

3.4. Level sets

The Hamiltonian function H' allows only bounded motions and almost all its orbits are concentrated in the corresponding orbits of the group g'. Therefore, the level set of any regular point is a disjoint union of finite number ($\geqslant 1$) of two-dimensional orbits (of g') diffeomorphic to tori. The phase-space is foliated with the finite number of two-parameter families of nested tori. In our case there are four such families denoted by w_i', $i = 1, \ldots, 4$ whose relation to the regular sets Q_i' is now described.

The set Q_1' corresponds to two families w_1' and w_2' and figure 4 shows the level set of one regular point from Q_1'. We are using the semiparabolic coordinates $x = (u^2 - v^2)/2$, $y = uv$. In these coordinates the phase-space is the quotient space of the space $\boldsymbol{R}^4 = \{(u, v, p_u, p_v)\}$ defined with the identification $(u, v, p_u, p_v) \equiv (-u, -v, -p_u, -p_v)$.

At this point we shall briefly discuss the Coulomb singularity of dynamical system defined by equation (9). As seen in figure 4, the regular points from the set Q_1' indeed correspond to two-dimensional tori. However, these tori actually belong to a regularized dynamical system which is formally obtained with the use of the coordinate transformation $(x, y) \to (u, v)$. This transformation is singular at the origin of the configurational space, i.e. for $x = y = 0$ (or $u = v = 0$). If we transform the tori from figure 4 back to the Cartesian

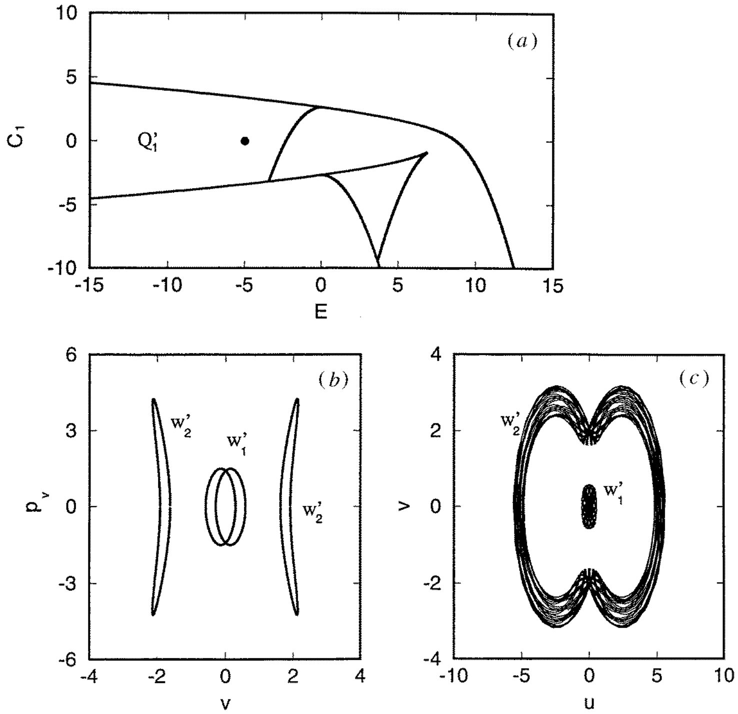

Figure 4. The level set of the regular point from the set Q'_1 for the case $f_s = 2.67$. (a) The location of the regular point. (b) The surface of section (SOS) of the level set defined with $u = 0$. (c) The projection, on the u–v plane, of two invariant tori w'_1 and w'_2.

coordinates, the resulting two-dimensional manifolds which correspond to nonregularized system will have singularities at the origin, i.e. they will not be diffeomorphic to tori. Therefore, all our statements about topology of the level sets hold only for the regularized system.

The tori from w'_1 and w'_2 are nested in such a way that they encircle one-dimensional orbits (of the group g') corresponding to the critical curve θ_1^+, figure 2(b). We now see that these orbits are stable periodic orbits of the Hamiltonian function H'.

Figures 5(b) and (d) show that each level set along the critical curve θ_1^- is a disjoint union of one one-dimensional orbit of the group g' (stable periodic orbit of H', equation (17), encircled with the tori of the family w'_1) and a torus from w'_2. Therefore the regular set Q'_2 corresponds only to the family of tori w'_2, figures 5(c) and (e).

Two families of tori w'_1 and w'_2 merge into each other along the critical curve ϵ^+, figure 6, and bifurcate into a new single family of tori w'_3 which corresponds to the regular set Q'_3, figure 8. The level set of each point from the critical curve ϵ^+ is separatrix i.e. it contains one closed one-dimensional orbit of the group g' and two two-dimensional noncompact orbits, both diffeomorphic to cylinders. The closed orbit, equation (20), is an unstable periodic orbit of H', and the entire separatrix is at the same time its stable and unstable manifold.

The family w'_2 bifurcates into family w'_3 also along the critical curve θ_2^-. Figure 7 shows an example of the separatrix (corresponding to a point from θ_2^-) which contains one unstable periodic orbit of H', equation (17), and one two-dimensional noncompact orbit of the group g' diffeomorphic to cylinder.

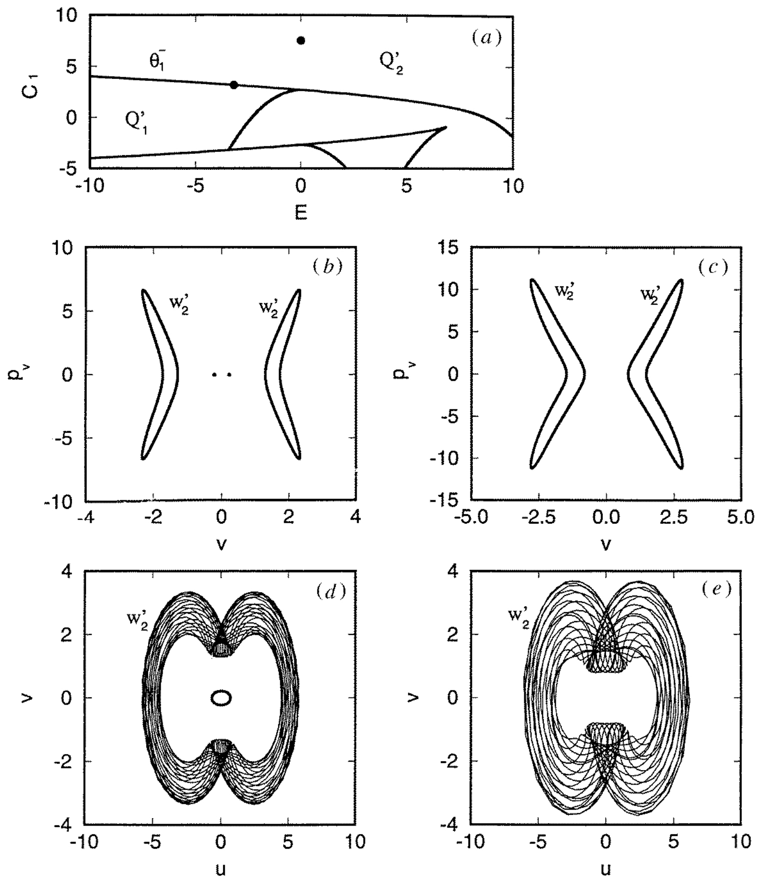

Figure 5. The same as in figure 4 but for the critical point from the curve θ_1^-, (b), (d), and for the regular point from Q_2', (c), (e).

The regular set Q_4' corresponds to two families of tori, w_3' and w_4', see figures 9(a), (c) and (e). These two families 'meet' along the critical curve θ_3^+ and form the separatices shown in figure 10. On the other hand, the level set of the point from the critical curve θ_4^+, figures 9(a), (b) and (d) are a disjoint union of a torus from w_3' and a stable periodic orbit of H', equation (17), encircled with the tori of the family w_4'.

Finally, the periodic orbits of H' corresponding to the critical curves θ_2^+, equation (17), and ϵ_1^-, equation (21), are encircled with the tori from the family w_3' while two orbits corresponding to ϵ_2^-, figure 3(c), are encircled, one with the tori from w_3' and another with the tori from w_4'. All of these orbits are therefore stable.

4. Phase-space structure of the three-dimensional system

In this section, we consider the topological structure of the full three-dimensional system H. In this case the set $Q = EM(P) = Q_r \bigcup Q_{cr}$ is a three-dimensional subset of the space $\boldsymbol{R}^3 = \{(E, C_1, C_2)\}$. The point $(E, C_1, C_2) \in Q$ is critical if the corresponding level set μ_{E,C_1,C_2} contains at least one critical orbit of the three-parameter group g (generated by H, H_1 and H_2). If the point (E, C_1, C_2) is regular then μ_{E,C_1,C_2} is a disjoint union

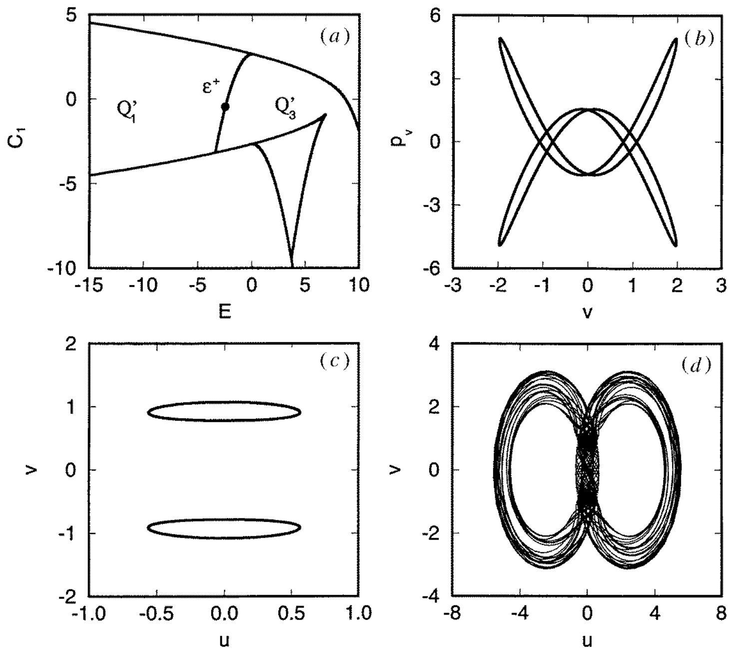

Figure 6. The level set of the critical point from the curve ϵ_+ for the case $f_s = 2.67$. (*a*) The location of the critical point. (*b*) The SOS of the level set defined with $u = 0$. (*c*) The projection, on the u–v plane, of the unstable periodic orbit which belongs to the level set. Two orbits should be identified because $(u, v, p_u, p_v) \equiv (-u, -v, -p_u, -p_v)$. (*d*) The projection, on the u–v plane, of the level set (separatrix).

of a finite number of three-dimensional smooth manifolds—orbits of the group g, which are diffeomorphic to the three-dimensional tori. Again, as in the previous section, all these statements actually hold for the regularized system. In the three-dimensional case, the regularization is more complicated and can be accomplished by using the Kustaanheimo–Stiefel transformation (Stiefel and Scheifele (1971)).

4.1. Critical points

The most convenient way to represent sets Q_r and Q_{cr} is to give plots of their intersections with various planes of the space $\boldsymbol{R}^3 = \{(E, C_1, C_2)\}$ defined by fixing the third integral C_2. Such intersections belong to (for given value of the parameter f_s) five different topological classes. The first class consists of only one element: it is the intersection of Q with the plane $C_2 = 0$. The second class of topologically equivalent sets contains the intersections of Q with the planes for which $0 < |C_2| \leqslant 1$, while the third, fourth and fifth classes contain the intersections of Q with the planes for which $1 < |C_2| < C_{2G}$ (where $C_{2G} = C_{2G}(f_s)$, see the first paragraph after equation (27)), $|C_2| = C_{2G}$ and $|C_2| > C_{2G}$, respectively. Again, like in the two-dimensional case the intersection of Q with the plane $C_2 = 0$ has two different structures depending on whether $f_s > (\frac{4}{9})^{2/3}$ or $f_s \leqslant (\frac{4}{9})^{2/3}$, while the topological structure of the intersections of Q with the planes $C_2 \neq 0$ does not depend on f_s. Note that the rank$(M(\boldsymbol{r}, \boldsymbol{p}))$, equation (7), corresponding to the Hamiltonian functions H, H_1 and H_2, equations (1)–(3), is invariant under the discrete transformation $(x, y, z, p_x, p_y, p_z) \to (x, y, -z, p_x, p_y, -p_z)$ while the Hamiltonian functions transform as $(H, H_1, H_2) \to (H,$

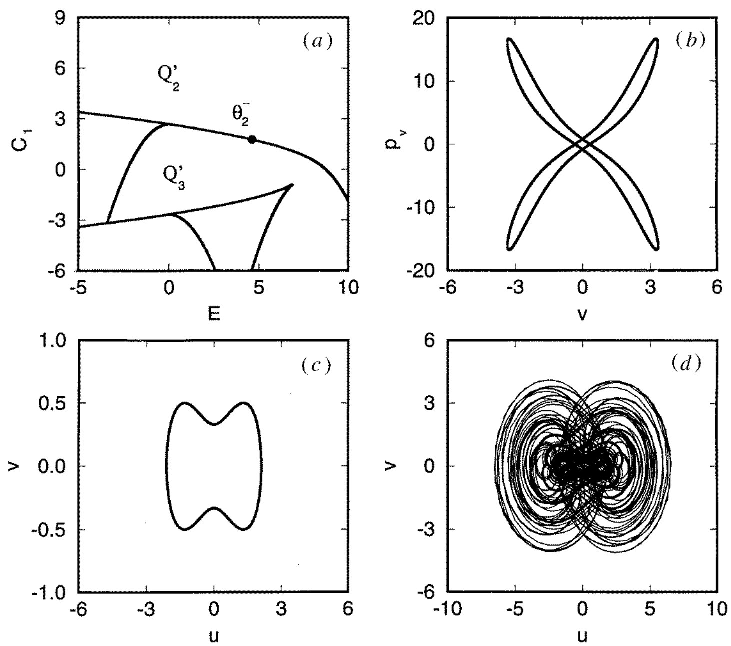

Figure 7. The same as in figure 6 but for the critical point from the curve θ_2^-.

H_1, $-H_2$). Therefore the sets Q, Q_r and Q_{cr} are all symmetric under transformation (E, C_1, C_2) $\to$ (E, C_1, $-C_2$). In figures 11(a)–(e) we give (for $f_s = 2.67$) five plots, each representing one class of topologically different intersections of Q with C_2 planes. The shaded regions in figures 11(a)–(e) do not belong to Q.

From figures 11(a)-(e) we see that the open set Q_r of regular points is not connected, it consists of two open connected components, i.e. $Q_r = Q_1 \bigcup Q_2$, one of which, Q_2, is not simply connected. As in the two-dimensional case, the set of critical points Q_{cr} decomposes, into several subsets of different dimensions. We distinguish four two-dimensional subsets—surfaces: $\sigma^+, \sigma^-, \epsilon^+$ and ϵ^-, 10 one-dimensional subsets—curves: θ_1^-, θ_2^-, θ_1^+, θ_2^+, θ_3^+, θ_4^+, μ, ν, α and β, and three points F_+, F_- and G. This decomposition is dictated by the topological structure of level sets, i.e. the level sets that correspond to the critical points of the same set are mutually diffeomorphic. Analogously, the level sets of the regular points which belong to the same connected component Q_1 or Q_2 are mutually diffeomorphic. From figure 11(a) we see that the two points F_+, F_- and the six curves θ_1^-, θ_2^-, θ_1^+, θ_2^+, θ_3^+ and θ_4^+ which all lie in the $C_2 = 0$ plane are the same as those in figure 1(a), only now the corresponding level sets are different. On the other hand, the other three curves from figure 1(a) ϵ^+, ϵ_1^- and ϵ_2^- are not distinguished any more, they are just subsets of the surfaces $\epsilon^+, \epsilon^- \subset Q_{cr}$.

4.2. Level sets of regular points

In contrast with the two-dimensional case, now only two three-parameter families of three-dimensional tori, denoted by w_1 and w_2, foliate the phase-space. The regular set Q_1 corresponds to both families. Similarly to w_1', the family of tori w_1 is 'bounded' by the critical surfaces σ^+, σ^-, ϵ^+ and the critical curves θ_1^+ and θ_1^-. On the other hand, the family of tori w_2 can be extended (similarly to w_2') across the critical surface σ^- and curve

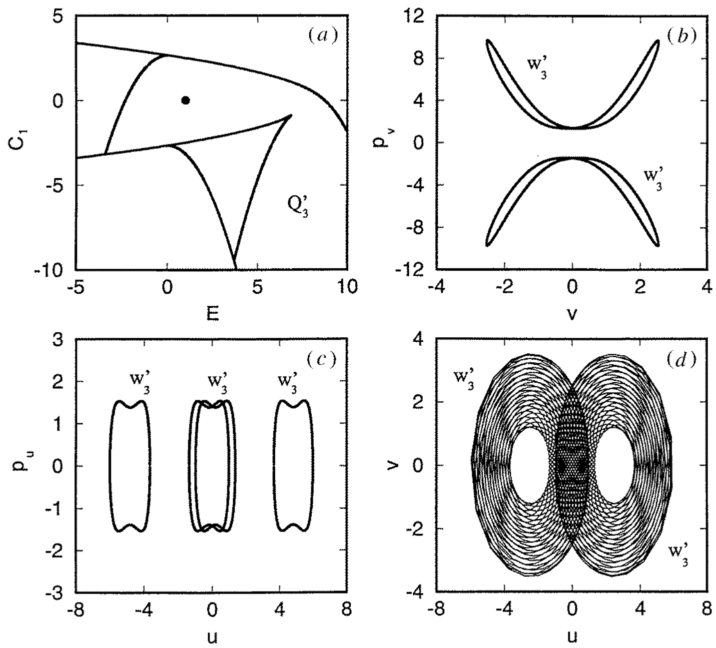

Figure 8. The level set of the regular point from the set Q_3' for the case $f_s = 2.67$. Two tori are identified. (a) The location of the regular point. (b) The SOS of the level set defined with $u = 0$. (b) The SOS of the level set defined with $v = 0$. (c) The projection, on the u–v plane, of the invariant tori w_3'.

θ_1^- into the open regular set Q_2. Therefore, Q_2 corresponds to the single family of tori w_2. Two families of tori w_1 and w_2 'meet' at the critical surface ϵ^+ and form separatrices.

The relation between the tori of the three-dimensional system with those of the reduced two-dimensional system is now described. The intersections of the tori from the families w_1 and w_2, which correspond to regular points from Q_1 and for which $C_2 = 0$, with the four-dimensional plane $z = 0$, $p_z = 0$ give two-dimensional invariant tori from families w_1' and w_2'. From figure 11(a) we see that the intersection of the set Q_2 with the plane $C_2 = 0$ decomposes into three disjoint open subsets which correspond to three regular sets Q_2', Q_3' and Q_4' in figure 1(a). Consider now the intersection of the torus from the family w_2, which corresponds to the regular point from Q_2 and for which $C_2 = 0$, with the four-dimensional plane $z = 0$, $p_z = 0$. If the regular point belongs to Q_2' or Q_3' the intersection gives two-dimensional invariant torus from the family w_2' or w_3'. If the point belongs to Q_4' the intersection gives two two-dimensional tori, one belonging to the family w_3' and another to w_4'.

Let W_1 and W_2 denote the unions of the tori from families w_1 and w_2, respectively. By definition they are open ($3n$-dimensional) subsets of the phase-space foliated by the corresponding families of tori. We see that because of the critical curves θ_2^-, θ_3^+, θ_4^+ and β the regular set Q_2 is not simply connected and therefore also the set W_2 is not simply connected. On the other hand, the sets Q_1 and W_1 are topologically trivial (i.e. diffeomorphic to $\boldsymbol{R}^3$ and $\boldsymbol{R}^6$ respectively). One of the consequences of this fact is that the family of tori W_1 admits the so-called global action-angle variables (Cushman and Dustermaat 1988) while such coordinates cannot be defined for the family of tori W_2.

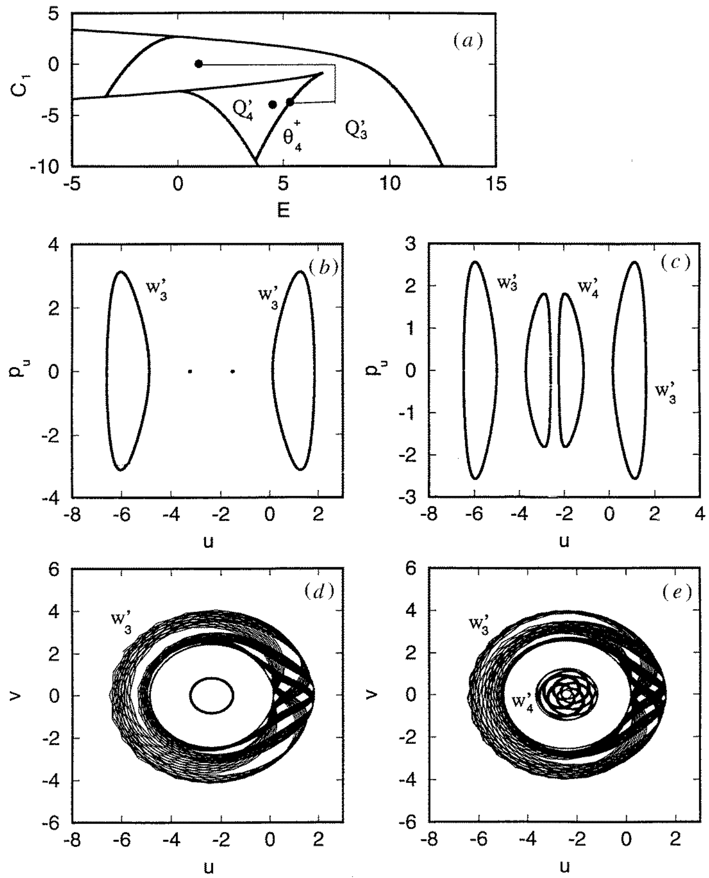

Figure 9. The same as in figure 8 but for the critical point from curve θ_4^+, (*b*), (*d*), and for regular point from Q_4', (*c*), (*e*).

4.3. Level sets of critical points $F_\pm$

The critical points F_+ and F_- correspond to only two (unstable) fixed points (denoted also by $F_\pm$) of all Hamiltonian functions H, H_1 and H_2, i.e. to zero-dimensional orbits of the group g. Their locations in the phase-space are given by equations (11), (12) and $z_{F_\pm} = 0$, $p_{zF_\pm} = 0$, while the corresponding values of the Hamiltonian functions H, H_1 and H_2 are given by equation (13) and $C_{2F_\pm} = 0$.

The easiest way to construct corresponding level sets is to act with the group g_2^t on the level sets of two-dimensional systems which correspond to points F_+ and F_-, see figures 2(*c*) and 3(*d*). It appears that not only F_+ but also all points in figures 2(*c*) are fixed under the action of the group g_2^t, i.e. under the motion of the Hamiltonian function H_2. Therefore, the level set of the critical point F_+ in the three-dimensional case coincides with that of the reduced two-dimensional system, i.e. it consists of one zero-dimensional orbit of the group g—unstable fixed point F_+ and two (noncompact) one-dimensional (critical) orbits of the group g diffeomorphic to $\boldsymbol{R}$.

On the other hand, the periodic orbit of the reduced system which belongs to the level set of the critical point F_-, figure 3(*d*), is not invariant under the action of the group g_2^t. The corresponding level set of the three-dimensional system is connected and

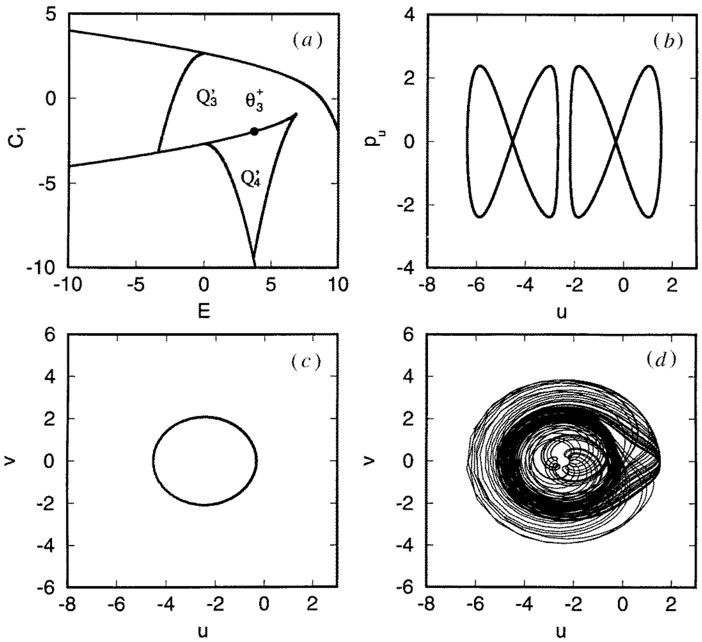

Figure 10. The same as in figure 6 but for the critical point from the curve θ_3^+.

consists of one zero-dimensional orbit of the group g—the unstable fixed point F_- and one (noncompact) two-dimensional (critical) orbit of the group g diffeomorphic to the two-dimensional cylinder. The level sets of critical points $F_\pm$ are, at the same time, stable and unstable manifolds of the corresponding fixed points.

4.4. Level sets of critical curves $\theta_i^\pm$

The level sets of the critical points which belong to six curves θ_1^-, θ_2^-, θ_1^+, θ_2^+, θ_3^+ and θ_4^+ contain one-dimensional critical orbits of the group g diffeomorphic to circles; they are periodic orbits of the Hamiltonian function H. These periodic orbits are in fact those of the reduced two-dimensional system defined in the previous section, see equations (14)–(17), i.e. they lie in the $z = 0$, $p_z = 0$ plane. This follows from the fact that the phase-space points belonging to these orbits are all fixed points of the group g_2^t, i.e. of H_2. Direct calculation shows that these are the only fixed points of H_2. This means that the level sets of critical points from curves θ_1^+ and θ_2^+ are the same as in the two-dimensional case, figures 2(*b*) and 3(*b*), i.e. they consist of stable periodic orbits (one-dimensional critical orbits of the group g).

Recall that, in the case of the reduced system, the level set of each point from the critical curve θ_1^- contains, besides the stable periodic orbit, two-dimensional torus of the family w_2', figures 5(*b*) and (*d*). The action of the group g_2^t on this torus generates the corresponding three-dimensional invariant torus of the family w_2. Hence, the level set of each critical point from the curve θ_1^- is disconnected, it consists of one one-dimensional critical orbit of the group g—stable periodic orbit of H, and one three-dimensional orbit—torus from the family w_2.

Consider now the level set of any given point from the critical curve θ_4^+. Here, the level set of the reduced system contains, besides the stable periodic orbit, two-dimensional torus,

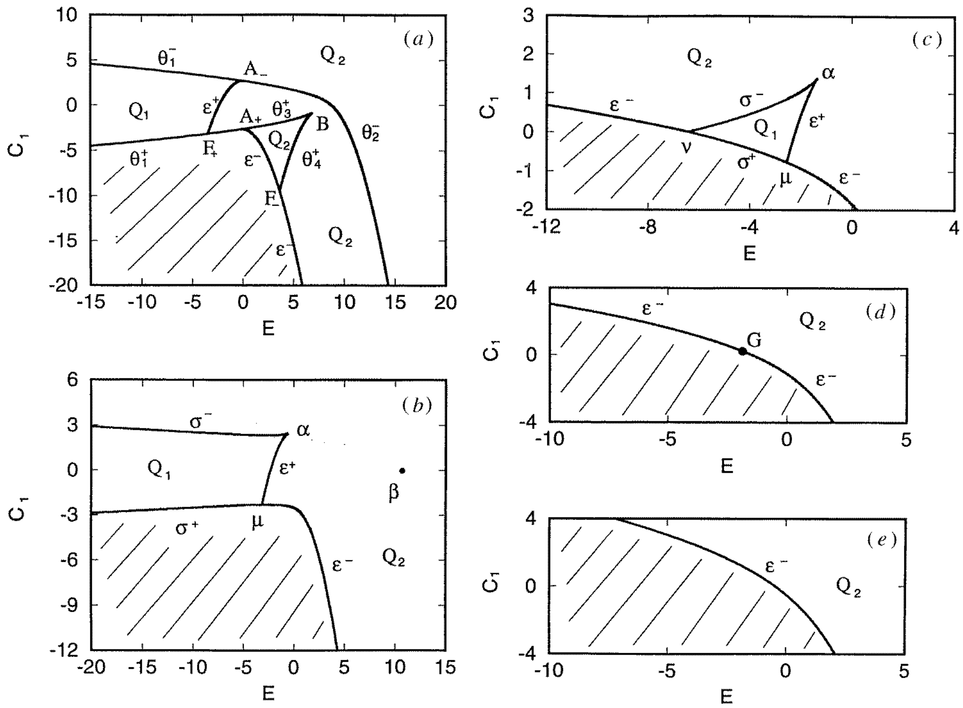

Figure 11. The intersections of the set Q and various C_2 planes for the case $f_s = 2.67$. (a) $C_2 = 0$, (b) $|C_2| = 0.5$, (c) $|C_2| = 1.5$, (d)$|C_2| = C_{2G} = 2.352$, and (e) $|C_2| = 3$.

now from the family w_3', see figure 9(d), and again the periodic orbit is invariant under the action of the group g_2^t. However, the action of the group g_2^t on the torus (of the family w_3') generates not a three-dimensional torus, but rather a three-dimensional noncompact orbit of the group g, which is diffeomorphic to the direct product of the two-dimensional torus and the real line (see equation (8)). The whole level set is connected and compact, and can be viewed as a compactification of noncompact three-dimensional orbit. The periodic orbit, is now unstable (in two-dimensional case it was stable) and the level set is its unstable and stable manifold.

Recall that the level sets of the reduced system corresponding to the points from the curves θ_2^- and θ_3^+ have different structures than those of the points from the curve θ_4^+, they are separatrices, see figures (7) and (10). Nevertheless, it appears that the action of the group g_2^t on two-dimensional cylinders that belong to separatrices generates noncompact three-dimensional orbits of the same structure as described above. Therefore, in the three-dimensional case the level sets of points from the curves θ_2^- and θ_3^+ have the same structures as those of the points from the curve θ_4^+, i.e. each such level set is connected and consists of one one-dimensional orbit of the group g (unstable periodic orbit of H), and one noncompact three-dimensional orbit of the group g diffeomorphic to the direct product of the two-dimensional torus and the real line.

In the case of critical curves μ, ν and β and the point G (see figures 11(a)–(d)), the level set corresponding to each point from them contains one closed one-dimensional critical orbit of g—the periodic orbit of all Hamiltonian functions H, H_1 and H_2. It appears that each such periodic orbit intersects the (five-dimensional) plane $z = 0$ of the phase-space P

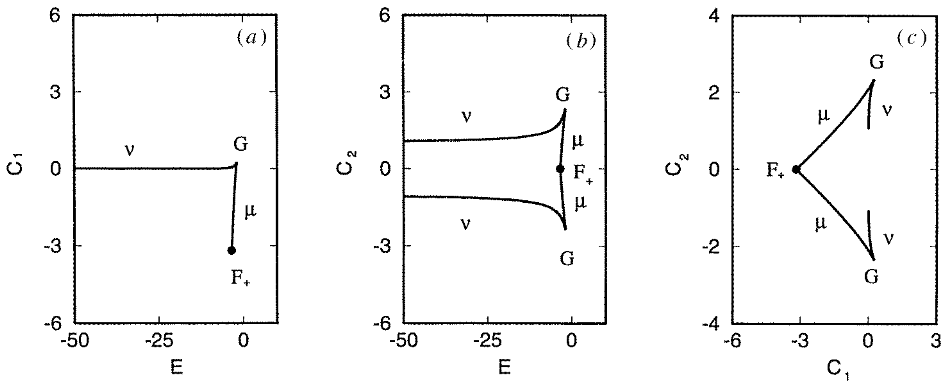

Figure 12. The plots of the critical curves μ and ν for the case $f_s = 2.67$. (a) Projection on the E–C_1 plane. (b) Projection on the E–C_2 plane. (c) Projection on the C_1–C_2 plane.

in exactly two points. Let $(\rho, \phi, 0, p_\rho, p_\phi, p_z)$ be the cylindrical coordinates of one of these intersecting points, then the coordinates of the other point are $(\rho, -\phi, 0, -p_\rho, p_\phi, -p_z)$. The coordinates of these intersecting points of periodic orbits corresponding to all critical curves μ, ν, β and the point G satisfy the following equations:

$$p_\phi = \rho^2 \qquad p_\rho = \tfrac{2}{3}\rho \tan\phi \qquad p_z^2 = \rho(\tfrac{4}{9}\rho - f_s\cos\phi)\tan^2\phi \tag{22}$$

while the corresponding values of the Hamiltonian functions are determined by:

$$E = \frac{4}{9}\frac{\rho^2}{\cos^2\phi} - \frac{8}{9}\rho^2 - \frac{1}{2}f_s\frac{\rho}{\cos\phi} + \frac{3}{2}f_s\rho\cos\phi - \frac{1}{\rho} \tag{23}$$

$$C_1 = \frac{3}{2}f_s^2\rho^2\cos^2\phi + f_s\left(1 + \frac{14}{9}\rho^3\right)\cos\phi + \frac{3}{2}f_s^2\rho^2 - \frac{4}{3}\rho - \frac{8}{27}\rho^4 2 f_s\frac{\rho^3}{\cos\phi} + \frac{16}{27}\frac{\rho^4}{\cos^2\phi} \tag{24}$$

$$C_2 = \tfrac{3}{2}p_z^3\cot\phi. \tag{25}$$

In addition, for the critical curve β it holds that

$$\cos\phi = \frac{-\frac{9}{4}f_s\rho^2 + \left(\frac{81}{16}f_s^2\rho^4 + 9\rho^3 - 4\rho^6\right)^{1/2}}{2\left(\frac{9}{4} - \rho^3\right)} > 0 \qquad 0 < \rho < \rho_{\max}$$
$$\rho_{\max} = (\tfrac{9}{4})^{1/3} \quad \text{if } f_s \geqslant (\tfrac{4}{9})^{2/3} \qquad \rho_{\max} = r_- \quad \text{if } f_s < (\tfrac{4}{9})^{2/3} \tag{26}$$

while for the curves μ and ν and for the point G one has

$$\cos\phi = \frac{-\frac{9}{4}f_s\rho^2 - \left(\frac{81}{16}f_s^2\rho^4 + 9\rho^3 - 4\rho^6\right)^{1/2}}{2(\frac{9}{4} - \rho^3)} < 0 \qquad 0 < \rho < r_+. \tag{27}$$

Using equations (22), (27) and (25), $|C_2|$ becomes a function of ρ. This function has an absolute maximum, and the point at which the maximum is reached corresponds to the critical point G, i.e. $\max(|C_2(\rho)|) = |C_2(\rho_G)| = C_{2G}$, while the critical curves ν and μ correspond to the following ranges of the coordinate ρ in equations (22)–(25) and (27)

$$0 < \rho < \rho_G \quad \text{for } \nu \qquad \rho_G < \rho < r_+ \quad \text{for } \mu. \tag{28}$$

In figure 12 we give the plots of the critical curves μ and ν, while in figure 13 we give the plots of the critical curve β, again for the case $f_s = 2.67$.

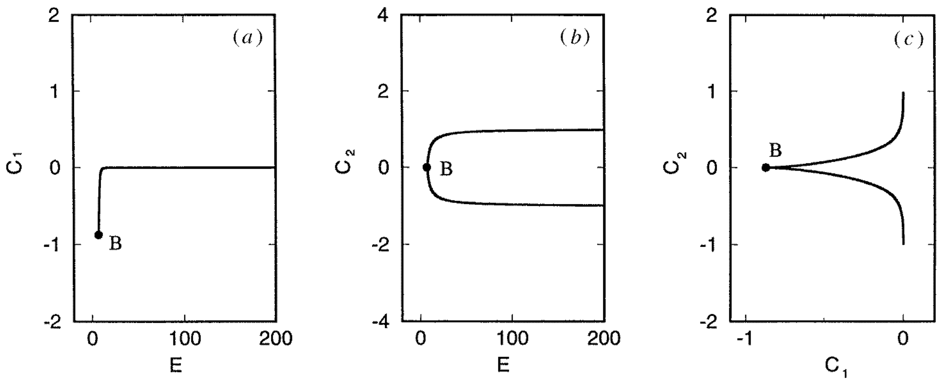

Figure 13. The same as in figure 12 but for the critical curve β.

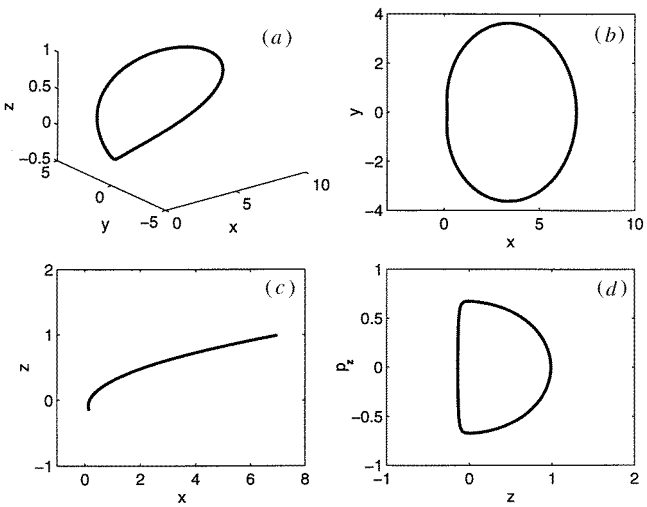

Figure 14. The plots of the unstable periodic orbit corresponding to the critical point from the curve β for which $E_1 = 7.78$, $C_1 = -0.35$, $C_2 = 0.13$ and $f_s = 2.67$. (a) The projection on the configurational space x–y–z. (b) The projection on the plane x–y. (c) The projection on the plane x–z. (d) The projection on the plane z–p_z.

From the above equations one finds for the curve β that if $\rho \to 0$ then $E \to +\infty$, $C_1 \to 0$ and $|C_2| \to 1$. Also when $\rho \to \rho_{\max}$ the critical curve β has B or F_- as the limiting point (see figure 13) depending on whether $f_s \geqslant (\frac{4}{9})^{2/3}$ or $f_s < (\frac{4}{9})^{2/3}$. For ν one finds that if $\rho \to 0$ then $E \to -\infty$, $C_1 \to 0$ and $|C_2| \to 1$. Also when $\rho \to r_+$ the curve μ has F_+ as the limiting point, see figure 12.

One way to construct a critical periodic orbit is to act on one of its intersecting points with the plane $z = 0$ (defined with the equations (22) and (26) or (27)) with any of the groups g_1^t, g_2^t or g_3^t. As an example we have given in figure 14 the plots of the periodic orbit corresponding to one of the critical points from the curve β.

We shall consider the level sets, corresponding to the critical curves μ and ν and the point G, together with the level sets of the critical surfaces. As for the curve β, the level sets of its points have the same structure as the level sets corresponding to critical curves θ_2^-, θ_3^+ and θ_4^+. Each such level set is connected and compact and, besides the critical one-dimensional orbit—unstable periodic orbit of H, see figure 14, contains only one three-dimensional orbit of the group g, which is therefore diffeomorphic to the direct product of the two-dimensional torus and the real line.

4.5. Level sets of critical surfaces $\sigma^{\pm}$, $\epsilon^{\pm}$

The families w_1 and w_2 of three-dimensional tori (corresponding to Q_1) have nested in such a way that they encircle two-dimensional orbits (tori) of the group g that correspond to the critical surface σ^+. Therefore σ^+ corresponds to two two-parameter families of two-dimensional stable invariant tori which we denote with w_1'' and w_2''. The tori from family w_1'' are nested themselves and they encircle one-dimensional orbits of the group g which correspond to the critical curve ν. At the same time the family of tori w_2'' continiously extends into the critical surface ϵ_-. Therefore, the level set of each point from the critical curve ν is disconnected and contains one stable periodic orbit (of the Hamiltonian function H) and one stable two-dimensional torus of the family w_2'', while the level set of each point from surface ϵ_- contains exactly one stable two-dimensional torus from family w_2''. (Recall that in the case of the two-dimensional reduced system of the previous section, the level set of each point from the critical curve ϵ_2^- consists of two stable periodic orbits, see figure 3(*c*). Obviously, these two periodic orbits are obtained by intersecting the corresponding torus from family w_2'' with the plane $z = 0$, $p_z = 0$, and the action of the group g_2^t on any of the periodic orbits generates the torus.)

The families w_1'' and w_2'' 'meet' along the critical curve μ to form a (two-dimensional) separatrix—the level sets which consist of one one-dimensional orbit of the group g—unstable periodic orbit (of H) and of two two-dimensional orbits of the group g diffeomorphic to cylinders.

From the structure of the level sets corresponding to the critical points from σ^+, ϵ^-, ν and μ one concludes that the level set of the critical point G contains one one-dimensional orbit of the group g—unstable periodic orbit of H, and one two-dimensional orbit of g diffeomorphic to cylinder.

The critical surface σ^- corresponds to yet another two-parameter family of two-dimensional tori which we denote with w_1'''. Again, the three-dimensional tori from the family w_1 are nested in such a way that they encircle two-dimensional tori from w_1''', while the family of tori w_2 continiously extends across the surface σ^- into regular set Q_2. The level set of each point from that surface is disconnected and contains one two-dimensional orbit of the group g—stable invariant torus from the family w_1''', and one three-dimensional orbit of g—invariant torus from the family w_2.

Two families of tori w_1 and w_2 'meet' at the critical surface ϵ^+ to form separatrices. The level set of each point from that critical surface is connected and contains one two-dimensional orbit of the group g—unstable invariant two-dimensional torus of H, and two three-dimensional orbits of the group g both diffeomorphic to the direct product of the two-dimensional torus and the real line. Clearly this level set also represents stable and unstable manifolds of the corresponding invariant unstable two-dimensional torus.

Finally, from the structure of the level sets corresponding to the points from the critical surfaces ϵ^+ and σ^-, it immediately follows that the level set of each point from curve α consists of one two-dimensional orbit of the group g—unstable two-dimensional torus of H, and of one three-dimensional orbit of the group g which is diffeomorphic to the direct product of the two-dimensional torus and the real line.

We conclude this subsection with the analytical formulae for the calculation of the critical surfaces. We have just seen that the level sets of critical points for which $C_2 \neq 0$

(and which do not belong to the curve β) contain certain two-dimensional critical orbits of the group g. It appears that each of these critical two-dimensional orbits intersects four-dimensional surfaces defined with the equations (in cylindrical coordinates)

$$z = 0 \qquad p_\rho^2 - \tfrac{3}{2} f_s p_\rho \sin\phi - p_z^2 = 0 \tag{29}$$

in a finite number of points. Let $X \subset P$ be the set of all these intersecting points. One can show that all the phase-space points which belong to X satisfy, besides above, the following relations:

$$p_\phi \sin\phi - 2 p_\rho \rho \cos\phi + \tfrac{1}{3}\rho^2 \sin\phi = 0 \tag{30}$$

$$3 p_\rho \cos\phi - 2\rho \sin\phi \neq 0 \qquad p_\rho - \tfrac{3}{2} f_s \sin\phi \neq 0 \qquad p_z \neq 0 \tag{31}$$

$$a_4(\rho,\phi) p_\rho^4 + a_3(\rho,\phi) p_\rho^3 + a_2(\rho,\phi) p_\rho^2 + a_1(\rho,\phi) p_\rho + a_0(\rho,\phi) = 0 \tag{32}$$

where

$$\begin{aligned}
a_4(\rho,\phi) = \rho^2(1 + 4\cos^2\phi + 3\cos^4\phi a_3(\rho,\phi)\\
&= -\tfrac{4}{3}\rho^3 \sin\phi \cos\phi (1 + \cos^2\phi)\\
&\quad -3 f_s \rho^2 \sin\phi (1 + \tfrac{3}{2}\cos^2\phi - \tfrac{1}{2}\cos^4\phi), a_2(\rho,\phi)\\
&= -2 f_s \rho^3 \sin^2\phi \cos^3\phi + \tfrac{9}{4} f_s^2 \rho^2 \sin^4\phi - 2\rho \sin^2\phi (1 + 2\cos^2\phi), a_1(\rho,\phi)\\
&= \tfrac{2}{3} f_s \rho^4 \sin^3\phi (1 + \cos^2\phi) + 3 f_s^2 \rho^3 \sin^3\phi \cos\phi\\
&\quad +\tfrac{4}{3}\rho^2 \sin^3\phi \cos\phi + 3 f_s \rho \sin^3\phi, a_0(\rho,\phi)\\
&= \sin^4\phi(-f_s^2\rho^4 + 1).
\end{aligned}$$

In other words, the level set of any critical point for which $C_2 \neq 0$ (and which does not belong to β) contains at least one point from the set X. Therefore, the energy-momentum map corresponding to three Hamiltonian functions, equations (1)–(3), maps the subset X of the phase-space P *onto* the set of all critical points which belong to the surfaces σ^+, σ^-, ϵ^+, ϵ^- and the curves μ and ν.

5. Application to hydrogen atoms in external fields

We now briefly discuss the relationship between our integrable system and some important physical systems of current interest. Recall that the Hamiltonian function (in atomic units) of the hydrogen atom in the CP field $H_{\mathrm{cir}} = \boldsymbol{p}^2/2 - 1/r + f(x\cos\omega t + y\sin\omega t)$, takes, in the frame rotating around the z-axis together with the field, the time-independent form (Chu (1978))

$$H_{\mathrm{cir}} = \frac{\boldsymbol{p}^2}{2} - \frac{1}{r} - \omega l_z + f x \tag{33}$$

where f and ω are the amplitude and the frequency of the applied field, respectively. The Hamiltonian function of the hydrogen atom in crossed magnetic and electric fields reads

$$H_{\mathrm{cross}} = \frac{\boldsymbol{p}^2}{2} - \frac{1}{r} + \frac{\gamma}{2} l_z + f x + \frac{\gamma^2}{8}(x^2 + y^2) \tag{34}$$

where γ and f are the strengths of the magnetic and electric fields, respectively. The above two systems have recently been studied (ionization process, oscillator strengths, transition to chaos,…) both experimentally [Bellermann *et al* (1994), Fu *et al* (1990), Wiebusch *et al* (1989)] and theoretically [Farrelly and Uzer (1995), Gourlay *et al* (1993), Marxer *et al* (1994), Milczewski *et al* (1994), Uzer and Farrelly (1995)].

From equations (1), (33) and (34) we see that the Hamiltonian functions $H_{\rm cir}$ and $H_{\rm cross}$ differ from that of the integrable system H only by a term proportional to ω^2 (in equation (34) we identify $\gamma = -2\omega$). Therefore, for small enough ω (as is the case in the microwave ionization experiments, Fu *et al* (1990)), one can try to use H as an approximation for $H_{\rm cir}$ and $H_{\rm cross}$. Simple analysis of the equations of motion generated with the functions $H_{\rm cir}$ and $H_{\rm cross}$ shows that the trajectories of these systems which are not located in the vicinity of nucleus are unbounded, while we have seen in section 4 that the Hamiltonian function H generates only bounded trajectories. However, the Hamiltonian functions $H_{\rm cir}$ and $H_{\rm cross}$ do generate bounded trajectories in the vicinity of the nucleus, and let $W_{\rm cir}$ and $W_{\rm cross}$ denote the regions of the phase-space filled with these bounded trajectories. Since the Hamiltonian functions $H_{\rm cir}$ and $H_{\rm cross}$ are not integrable, among bounded trajectories always exist irregular (chaotic) trajectories intertwining with regular trajectories. According to KAM theorem, Arnol'd (1978), the measure in the phase-space of the chaotic trajectories is proportional to the perturbational parameter. Now, it has been observed that for small ω, but already in a nonperturbative regime where most of the KAM–tori should break, the majority of bounded trajectories are still regular, i.e. they are confined to invariant three-dimensional tori. The existence of these invariant tori can be explained with the approximate integrals of motion which we recognize in the functions of equations (2) and (3).

From the above consideration we see that only those trajectories of the integrable system equation (1) which are located close to nucleus, i.e. which lie in open set W_1 (see section 4), can serve as approximations for the trajectories of $H_{\rm cir}$ and $H_{\rm cross}$. Indeed, for small enough ω, one can establish the correspondance between the invariant tori from the family w_1 and the invariant tori of $H_{\rm cir}$ and $H_{\rm cross}$, and in addition the open set W_1 can be used as a very good approximation for the regions $W_{\rm cir}$ and $W_{\rm cross}$.

There is another interesting realistic system which is even more closely related to our integrable case. Consider the hydrogen atom in the presence of the CP field together with the constant magnetic field orthogonal to the polarization plane. The Hamiltonian function (again in the rotating frame) reads:

$$H_{\mathrm{cir}\,\gamma} = \frac{\boldsymbol{p}^2}{2} - \frac{1}{r} - \left(\omega - \frac{\gamma}{2}\right) l_z + fx + \frac{\gamma^2}{8}(x^2 + y^2). \tag{35}$$

This system, especially its two-dimensional planar variant ($z = 0$, $p_z = 0$) has recently been studied in connection with the nondispersive wavepackets and atomic traps [Brunello *et al* (1996), Farrelly *et al* (1995), Lee *et al* (1995)]. Note that for the two-dimensional motion in the z-plane and in the special case when $(\omega - \gamma/2)^2/18 = \gamma^2/8$ (i.e. $\omega = \gamma/2 \pm 3\gamma/2$) the above Hamiltonian function reduces, after rescaling, to our two-dimensional case H', equation (9).

The discussion in this section suggests that the integrable system of equation (1) may be used in the study of the above mentioned more realistic systems. So far, we have applied the integrable system only in the calculation of the classical threshold fields for the ionization of the highly excited states of hydrogen atom by CP microwave fields in the two-dimensional [Raković and Chu (1994)] and three-dimensional [Raković and Chu (1995b)] cases. Excellent agreement with the experimental results [Bellermann *et al* (1994), Fu *et al* (1990)] has been achieved. Possible application of the integrable system to the other two realistic cases is currently under investigation

Acknowledgments

MJR acknowledges the support of the Kansas Institute for Theoretical and Computational Science through the K*STAR/NSF program. SIC is supported by National Science Foundation under the contract no PHY-95-12100.

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